GROTHENDIECK CONFERENCE
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TOËN: NONABELIAN HODGE STRUCTURES

Joint with Katzarkov, Pantev, Simpson, Vaquié, Vezzosi

1990: Simpson vision: There should exist nonabelian Hodge theory:
- Study of "Hodge structures" on "nonabelian/nonlinear" invariants of algebraic varieties / $\mathbb{C}$

Typical example

X smooth proj. / $\mathbb{C}$
$M_{DR}(X)$ = moduli space of flat bundles on X
We want to say that $M_{DR}(X)$ "has a Hodge structure"

Main questions
- What is a "Hodge structure" on an algebraic stack?
- What is a "Hodge structure" on an algebraic n-stack

(1) They should be such that they generalise the usual Hodge structures on vector spaces, and Hodge complexes. (2) Also, if X is as above, and F is a (n-stack with a) nonabelian Hodge structure, then H(X,F) should also be a nonabelian H.S.

2000 : Katzarkov, Pantev, Simpson :
Suggested definition of "NAMHS"
(nonabelian mixed Hodge structure).
Involves technical conditions, and point ②
above is not known for this construction.

2005 : New idea: Modify the above idea
to make it work.
Consider a point $x \in \mathcal{M}_{DR}(X)$ (a flat bundle)
Suppose x is a (real) VHS (variation of
pure Hodge structures). Then we expect
that $\hat{\mathcal{O}}_x$ should have a MHS..
"Hodge structure on $\mathcal{M}_{DR}(X)$" should be
something like a collection $\{\text{MHS on } \hat{\mathcal{O}}_x\}_x$
which is "continuous in x"

(only for those x corresponding to variations of Hodge structures)

Purpose of this talk:
Make the last sentence precise; in particular,
explain what we mean by $\hat{\mathcal{O}}_x$ for any algebraic
stack, and by "continuous in x"

"Definition": A nonabelian Hodge structure
is an algebraic n-stack F + some MHS
on $\{\hat{\mathcal{O}}_x\}_x$

Plan of talk

1) Algebraic n-stacks
2) Derived algebraic n-stacks
3) Definition of nonabelian Hodge str.
4) Theorem

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1) Algebraic n-stack

These are the coefficients for nonabelian cohomology.

An n-stack is a functor

$$F : \underline{\mathbb{C}\text{-alg}} \longrightarrow \underline{\mathrm{Top}} \quad (\text{or } \underline{s\mathrm{Set}})$$

with some conditions, including

- "sheaf condition"
- "$\pi_i(F(A)) = 0, \ \forall i > n$"

An n-stack is algebraic if it is a smooth quotient of an algebraic (n-1)-stack.
(Remark, I think (?): We could replace "smooth" here with fppf, and get an equivalent definition. Taking the étale topology would give Deligne-Mumford n-stacks)

We will restrict attention to n-stacks of finite type.

- If F is an algebraic n-stack of finite type, $\pi_0(F(\mathbb{C})) \simeq \coprod_{\text{finite}} M_i(\mathbb{C})$ where M_i is a variety$/\mathbb{C}$

- $x \in \pi_0(F(\mathbb{C}))$, $\pi_i(F, x)$ is an algebraic group $\forall i$

<u>Def</u>: An algebraic n-stack is called <u>special</u> if
- $\pi_1(F, x)$ is linear
- $\pi_i(F, x)$ is a vector space $\forall i > 1$

<u>Examples</u> (of (special?) algebraic n-stacks)
- algebraic variety $/\mathbb{C}$
- BG, where G = linear algebraic group $/\mathbb{C}$
- $K(\mathbb{G}_a, i)$: here $\pi_j(K(\mathbb{G}_a, i)) = \begin{cases} 0 & i \neq j \\ \mathbb{G}_a & i = j \end{cases}$

  (Remark: Could replace $\mathbb{G}_a$ with any (commutative?) group scheme, but then it would not be special unless the group scheme is a vector space)
- X finite CW 1-connected "$(X \otimes \mathbb{C})$" is an algebraic $\underset{(\infty)}{n}$-stack s.t. $\pi_i(X \otimes \mathbb{C}) \simeq \pi_i(X) \otimes \mathbb{G}_{a}$

* If F, F' are n-stacks, $\exists$ an n-stack $\underline{\mathcal{H}om}(F, F')$, which is algebraic under some conditions on F and F'

* Let X = algebraic var. $/\mathbb{C}$.
Have X_{top} as a constant stack
If F is an algebraic special n-stack (ASNS), then $H_B(X, F) := \underline{\mathcal{H}om}(X_{top}, F)$ is again an ASNS.

Two questions:

- $\widehat{\mathcal{O}}_x F$ for F an algebraic n-stack?
- Problem of controlling formal completion of $\underline{\mathcal{H}om}(X_{top}, F)$

These two problems are solved using the notions of derived n-stacks (i.e. derived algebraic geometry)

2) Derived algebraic n-stacks

References: Toen, Vezzosi : HAG I, II
Lurie : DAG I-V, ...

Original idea: (Kontsevich) "dg-variety" or "dg-scheme"

A derived stack (n-stack for some n) is a functor $F: \underline{\text{Comm DG-alg}} \longrightarrow \underline{\text{Top}}$ + conditions (descent, invariant by qis)

There is a natural notion of derived algebraic n-stack ($\exists$ natural definition of smooth and étale morphisms for DG things)

Compatibility: $\{\text{Alg. n-stacks}\} \subset \{\text{derived alg. n-stacks}\}$

fully faithfully

* A derived alg. n-stack has a "truncation" ("classical part") $t_0F \hookrightarrow F$ where t_0F is a (non-derived) algebraic n-stack

* $t_0F \hookrightarrow F$ is a "formal thickening"

* F comes equipped with a natural sheaf of dg-Lie algebras $\mathcal{L} \longrightarrow F$
  For a point $x \in F$, the fiber $\mathcal{L}_x$ determines the formal structure of F at x ($\mathcal{L}_x$ recovers F restricted to Artinian dg-algebras)
  E.g. $x \in$ scheme; then this is the tangent complex shifted by -1.

  (Remark: These $\mathcal{L}$ exist of course for non-derived stacks, but they behave well w.r.t. $\underline{Hom}$ in the derived setting)

Main Property

If F, F' are derived algebraic n-stacks, $\exists\ \mathbb{R}\underline{Hom}(F,F')$ a derived stack

$$\begin{array}{ccc} \mathbb{R}\underline{Hom}(F,F') \times F & \xrightarrow{ev} & F' \\ \downarrow{\scriptstyle p} & & \\ \mathbb{R}\underline{Hom}(F,F') & & \end{array} \qquad \mathcal{L}_{\mathbb{R}\underline{Hom}} = \mathbb{R}p_* \mathbb{L}ev^*(\mathcal{L}_{F'})$$

<u>Example</u>:

$K(\mathbb{G}_a; -i)$ as derived alg. n-stacks

$$\begin{gathered} K(\mathbb{G}_a; -i) \\ \text{ii} \\ \Omega_0 K(\mathbb{G}_a; -i+1) \\ \| \\ * \underset{K(\mathbb{G}_a; -i+1)}{\times} * \end{gathered}$$

<u>Note</u>: $K(\mathbb{G}_a; -1) = \Omega_0 \mathbb{A}^1$ is "Spec $\mathbb{C}[\varepsilon]$"
with $\deg(\varepsilon) = -1$

3) Nonabelian Hodge structures

<u>Simpson</u>: Rees construction

filtered vector spaces $\longleftrightarrow$ $QCoh(\mathcal{A})$
where $\mathcal{A} = [\mathbb{A}^1/\mathbb{G}_m]$

$\rightsquigarrow$ notion of a filtration on a derived algebraic stack F:

$\tilde{F} \to \mathcal{A}$ $\quad$ F derived alg.

$+\ \tilde{F}_{|\{1\}} \simeq F$ $\quad$ $\tilde{F}_{\{0\}} \circlearrowleft \mathbb{G}_m$ "assoc. graded"

<u>Def</u>: A (derived) nonabelian Hodge structure (NHS) of degree n:

* $F_{\mathbb{C}} \to \mathcal{A}$ a filtered derived alg. stack ("Hodge filtr.")
* $F_{\mathbb{Z}}$ a derived alg. stack $/\mathbb{Z}$
* a dg-Lie algebra $\mathcal{L}_{\mathbb{Z}} \to F_{\mathbb{Z}} \times \mathcal{A}_{\mathbb{Z}}$ ("weight filtration")

$$+ \quad \mathcal{L}_{\mathbb{Z}}|_{F_{\mathbb{Z}} \times \{1\}} \simeq \mathcal{L}_{F_{\mathbb{Z}}}$$

+ compatibility conditions
+ something being a Hodge complex

<u>Theorem</u> (not written down on blackboard)

The cohomology of a variety with coefficients in a derived NHS is again a derived NHS.

END

(<u>Remark</u>: In the above theory (where??), $\mathbb{C}$ can be replaced by any base ring. Need to replace commutative DG-algebras by simplicial comm. alg + other modifications)